Criterion of equilateral triangle.

https://www.linkedin.com/feed/update/urn:li:activity:6766265059908870144

Prove that the triangle ABC is equilateral if and only if

$$a\sin\left(A-\frac{\pi}{3}\right)+b\sin\left(B-\frac{\pi}{3}\right)+c\sin\left(C-\frac{\pi}{3}\right)=0.$$

Solution by Arkady Alt, San Jose, California, USA.

We have $\sum_{cyc} a\sin\left(A-\frac{\pi}{3}\right)=0 \Leftrightarrow \sum_{cyc} a\left(\sin A\cdot\frac{1}{2}-\cos A\cdot\frac{\sqrt{3}}{2}\right)=0 \Leftrightarrow$

$\sum_{cyc} a\sin A=\sqrt{3}\sum_{cyc} a\cos A \Leftrightarrow \frac{1}{2R}\sum_{cyc} a^2=\sqrt{3}\sum_{cyc} a\cdot\frac{b^2+c^2-a^2}{2bc} \Leftrightarrow$

$\frac{1}{R}\sum_{cyc} a^2=\frac{\sqrt{3}}{abc}\sum_{cyc} a^2(b^2+c^2-a^2) \Leftrightarrow \frac{1}{R}\sum_{cyc} a^2=\frac{\sqrt{3}}{4FR}\cdot 16F^2 \Leftrightarrow$

$\sum_{cyc} a^2=4\sqrt{3}\cdot F$ and since $16F^2=\sum_{cyc}(2b^2c^2-a^4)$ then

$\sum_{cyc} a^2=4\sqrt{3}\cdot F \Leftrightarrow \left(\sum_{cyc} a^2\right)^2=48F^2 \Leftrightarrow \left(\sum_{cyc} a^2\right)^2-48F^2=0 \Leftrightarrow$

$\sum_{cyc}(2b^2c^2+a^4)-3\sum_{cyc}(2b^2c^2-a^4)=0 \Leftrightarrow \sum_{cyc}(a^2-b^2)^2=0 \Leftrightarrow a=b=c$